On the History of Numerical Methods for Volterra Integral Equations

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1. Introduction

In this note we present a short historical account of the development of numerical methods for Volterra integral equations of the second kind, with the main part of the paper (Section 3) covering the period between about 1920 and the early 1960s. In order to see this development in its proper context we begin with a section in which we sketch the origins and some of the classical theory of Volterra integral equations, and we conclude by subsequently describing the principal areas of current research and a number of recent automatic computer codes.

The paper is concerned with linear (one-dimensional) Volterra integral equations of the second kind, i.e., equations in the (continuous) function y of the form

$$y(t) = g(t) + \int_0^t K(t,s)y(s)ds, \quad t \in I := [0,T], \tag{1.1}$$

and

$$y(t) = g(t) + \int_0^t (t-s)^{-\alpha} y(s)ds, \quad t \in I,\ 0<\alpha<1. \tag{1.2}$$

Their nonlinear counterparts are

$$y(t) = g(t) + \int_0^t k(t,s,y(s))ds, \quad t \in I, \tag{1.3}$$

and

$$y(t) = g(t) + \int_0^t (t-s)^{-\alpha} k(t,s,y(s))ds, \quad t \in I,\ 0<\alpha<1. \tag{1.4}$$

It will be assumed that the kernels $K(t,s)$ and $k(t,s,y)$ are given continuous function of their respective variables and (in the nonlinear case) are such that there exists a unique solution $y \in C(I)$ whenever the given function g is in $C(I)$. (Generalizations to, e.g., the L_2-setting are, of course, possible but will not be considered here.)

2. A SHORT HISTORY OF VOLTERRA INTEGRAL EQUATIONS

The classical papers of ABEL [1], [2] and of VOLTERRA [89] deal with the 'inversion of definite integrals': if g and G are given functions of one and two variables, respectively, find a (continuous) function y satisfying the first-kind integral equation

$$\int_0^t (t-s)^{-\alpha} G(t,s) y(s) ds = g(t), \quad t \in I, \ 0 \leqslant \alpha < 1 . \tag{2.1}$$

ABEL investigated the special case $G(t,s) \equiv 1$, $0 < \alpha < 1$, and derived the explicit inversion formula,

$$y(t) = \frac{\sin(\alpha\pi)}{\pi} \frac{d}{dt} \left(\int_0^t (t-s)^{\alpha-1} g(s) ds \right) . \tag{2.2}$$

He shows that equation (2.1), with $G(t,s) \equiv 1$, describes the problem of determining the equation of a curve in a vertical plane such that the time taken by a mass point to slide, under the influence of gravity, along this curve from a given positive height to the horizontal axis is equal to a prescribed (monotone) function of the height.

The general case was treated by VOLTERRA [89] in 1896: he showed, both for $\alpha = 0$ and for $\alpha \in (0,1)$, that if $G(t,t) \neq 0$ for all $t \in I$ and if g and G satisfy some obvious regularity conditions, then (2.1) can be rewritten as a second-kind integral equation (1.1) whose kernel is continuous on the domain $S := \{(t,s) : 0 \leqslant s \leqslant t \leqslant T\}$. Picard's method of successive approximations (proposed in his paper [72] of 1890) can then be employed; it leads, by means of the iterated kernels:

$$K_n(t,s) := \int_s^t K(t,v) K_{n-1}(v,s) dv \quad (n \geqslant 2), \ K_1(t,s) := K(t,s) ,$$

associated with $K(t,s)$ in (1.1), and the corresponding Neumann series:

$$R(t,s) := \sum_{n=1}^{\infty} K_n(t,s), \quad (t,s) \in S \tag{2.3}$$

(which, for $K \in C(S)$, converges absolutely and uniformly on S), to the 'inversion formula'

$$y(t) = g(t) + \int_0^t R(t,s) g(s) ds, \quad t \in I . \tag{2.4}$$

This inversion formula is no longer explicit, in the sense of (2.2), since the

resolvent kernel $R(t,s)$ cannot, in general, be found analytically.

We mention in passing that one of the tools used in the above convergence analysis is Dirichlet's formula (dating from 1837; cf. [26]) which states that

$$\int_0^a dt \int_0^t \Phi(t,s)ds = \int_0^a ds \int_s^a \Phi(t,s)dt ,$$

provided Φ is a continuous function. (This result was generalized in 1908 by HURWITZ [49] to include integrands containing weakly singular terms like $t^{\lambda-1}(t-s)^{\mu-1}$, with $\lambda>0$, $\mu>0$.)

Particular cases of the second-kind integral equation (1.1) occur already in the papers by POISSON [73] of 1826, where the kernel $K(t,s)$ is of convolution type (i.e., depends only on the difference $t-s$ of its arguments t and s), and by LIOUVILLE [58] of 1837. LIOUVILLE seems to have been the first to employ the idea of successive approximations in an integral equation, thus anticipating Picard's suggestion by some fifty years: in loc.cit. he applied it to the integral equation he had obtained by rewriting the initial-value problem for a second-order differential equation, and he so established the uniform convergence of the resulting sequence of approximants. (His idea was subsequently extended to ordinary linear differential equations of arbitrary order by CAQUÉ in 1864; cf. [14] for bibliographical details.) As far as the general integral equation (1.1) is concerned, one finds the approach used by VOLTERRA already in the thesis by LE ROUX in 1894 (published as [56] a year later); however, LE ROUX did not investigate the uniform convergence of the resulting Neumann series.

By the turn of the century the classical quantitative theory of linear Volterra integral equations with regular kernels had essentially been established. Later work on second-kind integral equations by EVANS [31] in 1910/11 concerning various types of singular kernels, by ANDREOLI [3] in 1914 concerning equations whose upper limit of integration, t, is replaced by some function $\Phi(t)$, and by LALESCO, SCHMIDT, and others at about 1908 (see [22], [42], [25] for details) was already overshadowed by the fundamental work by FREDHOLM in 1900, 1903 ([33]) and by HILBERT in 1904-1910 ([43]). The latter work on second-kind integral equations with fixed limits of integration marks the birth of functional analysis. (Compare the recent studies by MONNA [68] and by DIEUDONNÉ [25]; see also [30].)

3. EARLY NUMERICAL METHODS

The idea of replacing the integral in (1.1), with $t=t_n:=nh$ $(n=1,...,N;\ Nh=T)$, by a finite sum (i.e., by some quadrature formula), thus obtaining, in a recursive way, approximations $\{y_n\}$ to the exact values $\{y(t_n)\}$, was introduced by VOLTERRA in [89, pp. 219-220] and, more explicitly, in [90, pp. 40-45]. Setting

$$y_n = g(t_n)+h\cdot\sum_{j=0}^{n-1} K(t_n,t_j)y_j,\quad n=0,...,N \tag{3.1}$$

he obtained a linear system in $\mathbb{R}^{N+1}$ for $u_N := (y_0,y_1,...,y_N)^T$. (Note that the

right hand side corresponds to a particular Riemann sum, in which the values of the integrand are taken at the left endpoints of the subintervals $[t_j,t_{j+1}]$, $j=0,...,n-1$.) The system (3.1) has the form $(I_N-hA_N)u_N=g_N$. Here, I_N is the identity matrix, A_N is a strictly lower triangular matrix whose nontrivial elements are $K(t_n,t_j)$ $(0\leqslant j<n\leqslant N)$, and $g_N:=(g(t_0),...,g(t_N))^T$. Due to the special choice of the quadrature approximation the matrix I_N-hA_N is always nonsingular (for more general quadrature formulas this will hold only for sufficiently large values of N), and hence (3.1) is uniquely solvable. VOLTERRA employed this approach not for the actual numerical solution of (1.1) but to establish, by 'passing from finiteness to infinity' (as he called it), the identities

$$\begin{aligned} R(t,s) &= K(t,s)+\int_s^t K(t,v)R(v,s)dv \\ &= K(t,s)+\int_s^t R(t,v)K(v,s)dv, \quad (t,s)\in S\ , \end{aligned}$$

(now generally referred to as Fredholm identities) between the kernel $K(t,s)$, of (1.1) and the corresponding resolvent kernel $R(t,s)$ introduced in (2.3).

3.1 Related fields

Before we start discussing the early contributions to the numerical solution of integral equations, beginning with WHITTAKER' s paper [92] of 1918, we shall recall briefly what was known at that time in two fields closely related to numerical analysis, namely numerical integration (or quadrature) and the numerical solution of initial-value problems for ordinary differential equations. (The reader is referred to GOLDSTINE [36], GAUTSCHI [35], MILNE [67], and to the forthcoming monograph [40] by HAIRER, NØRSETT and WANNER for bibliographical and historical details.)

Numerical quadrature. The origins of numerical quadrature date back to the work of CAVALIERI (1639), GREGORY (1670), NEWTON (1676), COTES (1722), MACLAURIN (1742), SIMPSON (1743), and EULER (1755). The results of GAUSS (1814) on more general quadrature formulas were extended by JACOBI (1826) (who based his theory on the theory of orthogonal polynomials), and by CHRISTOFFEL (1852). The work of LOBATTO and RADAU on quadrature formulas possessing a certain number of prescribed abscissas (either both, or one of the endpoints of the interval of integration) dates from 1852 and 1880 respectively. Finally, the classical result on the integral representation of the error of a given quadrature formula, i.e. Peano's kernel theorem, was published in 1914.

Among the classical quadrature formulas it was the one known as Gregory's

rule which, as will be seen below, played initially the dominant role in the numerical solution of Volterra integral equations Gregory's rule in an extension of the trapezoidal rule, it has the form

$$\int_0^{t_n} f(s)ds \approx h\left[\tfrac{1}{2}f(t_0)+f(t_1)+\ldots+f(t_{n-1})+\tfrac{1}{2}f(t_n)\right] - \tag{3.2}$$

$$- h\cdot\sum_{i=1}^{q} c_i\left[\nabla^i f(t_n)+(-1)^i\Delta^i f(t_o)\right],$$

where $t_k := t_0+kh$ $(h>0)$, $t_0 := 0$, $\Delta^0 f(t_k) := f(t_k)$, $\Delta f(t_k) := f(t_{k+1})-f(t_k)$, $\Delta^i := \Delta(\Delta^{i-1})$ (with analogous definitions for the backward difference operator $\nabla f(t_k) := f(t_k)-f(t_{k-1})$), and where the first few coefficients c_i in the end corrections are given by $c_1 = 1/12$, $c_1 = 1/24$, $c_3 = 19/720$, $c_4 = 3/360,\ldots$, q is a given integer. Notice that the case $q=0$ corresponds to the trapezoidal rule. The above quadrature formula (3.2) is closely related to the Euler-MacLaurin summation formula (established some seventy years after Gregory's formula),

$$\int_0^{t_n} f(s)ds = h\left[\tfrac{1}{2}f(t_0)+f(t_1)+\ldots+f(t_{n-1})+\tfrac{1}{2}f(t_n)\right] -$$

$$-\sum_{k=1}^{m}\frac{h^{2k}B_{2k}}{(2k)!}\left[f^{(2k-1)}(t_n)-f^{(2k-1)}(t_0)\right] + R_m(f),$$

where $R_m(f) := -t_n h^{2m+2} B_{2m+2} f^{(2m+1)}(\xi)/(2m+2)!$ $(\xi\in[t_0,t_n])$; here, the B_j are the Bernoulli numbers (i.e., the coefficients of $t^j/j!$ in the power series expansion of $t/(e^t-1)$). Gregory's rule is obtained by using appropriate finite-difference approximations to the derivatives of f at the endpoints t_0 and t_n, followed by suitable truncation (compare also KRYLOV [54, pp. 35-38]).

Initial-value problems. In the development of numerical methods for the initial-value problem $y' = f(t,y)$, $y(t_0)=y_0$, an idea first encountered in Euler's work (1768) was used by GAUCHY in 1840 to derive a viable algorithm (now generally known as Euler's method), $y_{n+1} := y_n + hf(t_n,y_n)$ $(n\geq 0)$. Among the successors of this method are the one-step methods of RUNGE (1895), HEUN (1900), and KUTTA (1901) one of whose explicit four-stage, fourth order methods was, until fairly recently, simply referred to as *the* Runge-Kutta method. The (explicit) linear multistep methods known as Adams-Bashforth methods which were introduced by BASHFORTH and ADAMS in 1883 can also be considered as successors to Euler's method. The analogous implicit linear multistep methods (the methods of Adams-Moulton) originate from the work of MOULTON in the 1920s. Nyström's method for approximating the solution to the initial-value problem for a second-order differential equation dates from 1926. Except for some earlier surveys, the books by COLLATZ [21] and by MILNE [67] (whose first editions were published in 1951 and 1953 respectively) represent the first comprehensive accounts of numerical methods for ordinary differential equations.

Returning to Volterra integral equations and to the paper [92] by WHITTAKER of 1918, we observe that his methods for equations of the form (1.1) with convolution kernel $K(t,s)=a(t-s)$ do not yet reflect the fact that (1.1) may be viewed as a generalization of the initial-value problem for an ordinary differential equation, for whose numerical solution one might try to use suitable analogues of certain known methods for ordinary differential equations. Whittaker's first two methods are based on the assumption that the kernel $a(z)$ is given in the form of a numerical table. Using De Prony's method of 1795, he approximates $a(z)$ by an interpolant which is a linear combination of exponential functions; its construction involves the solution of a certain nonlinear algebraic equation. The approximation $u(t)$ to the exact solution is then of the form

$$u(t) = g(t)+\int_0^t r(t-s)g(s)ds, \quad t\in I ,$$

where $r(z)$ (which may be regarded as an approximation to the resolvent kernel associated with the kernel $a(z)$; cf. (2.4)) is again a linear combination of exponential functions whose exponents are given by the roots of the above nonlinear equation and whose coefficients depend on these roots as well as on the exponents occurring in the interpolant of $a(z)$. In the second method the interpolant is a polynomial, while in the third method it is assumed that one knows the Taylor expansion of $a(z)$; this then permits the computation of the Taylor expansion of the resolvent kernel. In both cases, the exact solution is approximated by an expression of the form (3.2).

3.2. The methods of Prasad

Multistep methods. The methods proposed by PRASAD [75] in 1924 are the true ancestors of most of the present-day numerical methods for (1.1) and (1.3); moreover, they are applicable not only to integral equations with convolution kernels but to general equations as well. His linear multistep method is based on the Gregory rule (3.2); for (1.3) it thus assumes the form

$$y_n = g(t_n)+h\sum_{j=0}^{n} w_{n,j}k(t_n,t_j,y_j), \quad n = q+1,...,N , \tag{3.3}$$

where $t_n := nh$ $(n=0,...,N;\ Nh=T)$, and where the weights $\{w_{n,j}\}$ are easily obtained from the coefficients characterizing Gregory's rule (3.2). PRASAD employs the value $q=4$; in addition to $y_0=g(0)$ he thus needs the values y_1,y_2,y_3,y_4 to start the recursion (3.3). These starting values may be obtained by means of the trapezoidal rule and Simpson's rule, possibly using smaller initial sub-intervals in order to attain sufficiently accurate approximations.

Runge-Kutta methods. In order to avoid methods that depend on starting values, PRASAD shows how the ideas of RUNGE, KUTTA and others can be adapted to generate approximations at $t_n = nh$ to the solution of (1.3). His starting point is one of the explicit four-stage, fourth-order methods introduced by KUTTA in 1901: in current notation (i.e., in terms of the so-called Butcher array: see, e.g., [40]) this method is characterized by

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array} := \begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ \frac{2}{3} & -\frac{1}{3} & 1 & 0 & 0 \\ 1 & 1 & -1 & 1 & 0 \\ \hline & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{array} .$$

Hence, for $t \in [t_n, t_{n+1}]$ the integral equation (1.3) is discretized by setting

$$y_{n+1} = \tilde{F}_n(t_{n+1}) + h \cdot \sum_{i=1}^{m} b_i k(t_{n+1}, t_n + c_i h, Y_{n,i}) , \tag{3.4a}$$

where the $Y_{n,i}$ are obtained from

$$Y_{n,i} = \tilde{F}_n(t_n + c_i h) + h \cdot \sum_{j=1}^{i-1} a_{i,j} k(t_n + c_i h, t_n + c_j h, Y_{n,j}) , \tag{3.4b}$$

$$(j = 1, \ldots, m) .$$

Here, $m = 4$, and $\tilde{F}_n(t)$ denotes a suitable approximation to the lag term

$$F_n(t) := g(t) + \int_0^{t_n} k(t, s, y(s)) ds, \quad t \in [t_n, t_{n+1}] , \tag{3.5}$$

of the equation (1.3), and the numbers b_i, c_i, and $a_{i,j}$ are the elements of the vectors b, c and of the matrix A, respectively, in the above array. PRASAD however, dismisses Runge-Kutta methods of the form (3.4) as being 'not so good as' the method based on Gregory's rule; they found a renewed interest only some thirty years later.

We note in passing the the first 'practical' application of a method of the form (3.3) (involving Gregory's rule (3.2) with $q = 0$) seems to occur in the book [20] by CARSON (pp. 145-146) in 1926; the method is employed to solve numerically a linear Volterra integral equation with convolution kernel found in the theory of electric circuits.

3.3. Convergence analysis

The first convergence analysis for quadrature methods (3.3) applied to the linear equation (1.1), exhibiting the relation between the errors of the underlying quadrature formula and the resulting order of the approximation error $e_n := y(t_n) - y_n$ of the method, was given by MIKELADZE [65] in 1935. The main tool in his analysis is a discrete version of Gronwall's inequality,

$$z_n \leqslant hC_1 \cdot \sum_{i=0}^{n-1} z_i + C_2, \quad n = 0,...,N ,$$

with $z_i \geqslant 0$, $C_1 > 0$, $C_2 > 0$ (see also below, Section 4). The methods studied by MIKELADZE are essentially those based on the various Gregory rules. The author's motivation for considering these methods lies in the numerical solution of higher-order linear differential equations: he suggests their being rewritten as Volterra integral equations of the second kind. An analogous idea is also used for second-order linear partial differential equations, here, the resulting integral equations contain double integrals.

The paper by KRYLOV [54] of 1949 deals also with the quadrature method (3.3) employing the Gregory rule (in the Russian literature this rule is often called the Euler-Laplace formula). Moreover, KRYLOV introduces block methods for the simultaneous computation of the starting values y_1, y_2 (if $q = 2$), or y_1, y_2, y_3, y_4 (if $q = 4$) needed in (3.3). These starting methods are obtained by choosing sets of q quadrature formulas of the same length ($q+1$ abscissas) and with the same degree of precision. (Compare also Wolkenfelt's thesis and his paper [94] for these and other starting methods.) These block methods involve kernel values $K(t,s)$ or $k(t,s,y)$ for $s > t$ which have to be found by a suitable extrapolation procedure; moreover, a linear (or, in the case (1.3), a nonlinear) system in $\mathbb{R}^2$ or in $\mathbb{R}^4$ has to be solved. Although these starting methods are chosen so that their orders of accuracy are consistent with the orders of accuracy of Gregory rules underlying (3.3), there is no convergence analysis. MIKELADZE' s paper [65] of 1951 takes up the ideas of Krylov and suggests a number of marginal improvements in the implementation of the methods.

In the early 1950s, explicit Runge-Kutta methods of the form (3.4) were considered once more, namely in the paper [84] by SUYAMA and NAKAMORI. Here, the interest was focused on the derivation of the so-called order conditions which the parameters c_i, b_i, and $a_{i,j}$ in (3.4a), (3.4b) have to satisfy if the method is to have (local) order $p = 4$. In other words, suppose that the lag term approximations $\tilde{F}_n$ in (3.4a) and (3.4b) are replaced by the *exact* lag term F_n introduced in (3.5) (i.e., the given integral equation (1.3) is solved *locally* on $[t_n, t_{n+1}]$, by assuming that $y(t)$ be known exactly on $[0, t_n]$); let the resulting approximation at $t = t_{n+1}$ be denoted by $\bar{y}_{n+1}$. What algebraic equations do the parameters c_i, b_i, and $a_{i,j}$ have to satisfy in order that $|y(t_{n+1}) - \bar{y}_{n+1}| \leqslant Ch^p$, where C is a constant not depending on h ? These order conditions are derived by Taylor expansion techniques (compare also [18] for a more elegant approach which is based on certain concepts from

graph theory and which extends the analogous theory of Butcher for Runge-Kutta methods in ordinary differential equations). However, questions regarding the convergence or the practical implementation of (3.4) (including the problem of how to generate suitable lag term approximations $\tilde{F}_n$) are not touched upon.

The first systematic convergence analysis for explicit Runge-Kutta methods (3.4) is due to POUZET and may be found in his thesis of 1962 and in [74]; see also GAMONDI and ITALIANI [34] for closely related results. POUZET showed that if the lag term approximations $\tilde{F}_n$ are based on quadrature formulas characterized by the weights b_i and the abscissas $t_k + c_i h$ $(i = 1,...,4;\ k < n)$, and if the Runge-Kutta method (3.4) has local order $p = 4$, then the approximation error $e_n := y(t_n) - y_n$ satisfies $\max_{(n)} |e_n| = \mathcal{O}(h^p)$, as $h \downarrow 0$, $Nh = const.$ Analogous results hold for other, suitably chosen lag term approximations.

A different class of explicit Runge-Kutta methods for (1.3) was introduced by BEL'TYUKOV [9] in 1965. Here, the underlying arrays of parameters are no longer those given by a Runge-Kutta method for a differential equation. We now have, instead of (3.4),

$$y_{n+1} = \tilde{F}_n(t_{n+1}) + h \cdot \sum_{i=1}^{m} b_i k(t_n + d_i h,\ t_n + c_i h,\ y_{n,i}) ,$$

with

$$y_{n,i} = \tilde{F}_n(t_n + c_i h) + h \sum_{j=1}^{i-1} a_{i,j} k(t_n + d_j h,\ t_n + c_j h,\ y_{n,j}) \ (j = 1,...,m) .$$

BEL'TYUKOV analyzed the methods corresponding to $m \leqslant 3$ and satisfying the conditions $d_i \geqslant c_i$ for all values of i. Even though these methods require fewer kernel evaluations than the methods (3.4) studied by POUZET it turns out (cf. [18]) that the construction of higher-order methods (of order $p \geqslant 4$) is quite difficult; in particular, there does not exist an explicit Bel'tyukov method with $p = m = 4$ (recall that, as shown by POUZET $p = m = 4$ is possible for (3.4)).

As far as high-order methods of Runge-Kutta type are concerned, we note that SCHOEDON [80] in 1970 studied a class of such methods based on certain Hermite quadrature formulas.

Returning briefly to linear multistep methods of the form (3.3), we point out the paper [52] by JONES of 1961: this paper contains a detailed convergence analysis of the trapezoidal method when applied to second-kind integral equations with convolution kernels (or to systems of such equations). Later analyses of linear multistep methods (3.3) were largely influenced by the fundamental work of DAHLQUIST and HENRICI on linear multistep methods for ordinary differential equations (dating from the late 1950s and the early 1960s). The first papers to extend their theory to Volterra integral equations are due to SPOHN [83] (1965) and to KOBAYASI [53] (1966).

Up to the early 1960s, almost no attention had been paid to the numerical solution of Volterra integral equations (1.2) and (1.4) whose kernels contain a weak (integrable) singularity of the form $(t - s)^{-\alpha}$, with $0 < \alpha < 1$. PRASAD [75,

p. 58] briefly mentions the possibility of rewriting (1.2), with rational $\alpha = p/q$ ($p,q \in \mathbb{N}$; p and q coprime), as an equation with regular kernel; the underlying change of variable implies that the upper limit of integration in (1.1) now becomes $t^{1/q}$. However, he gives no further details. WAGNER [91] in 1954 seems to be the first source suggesting in detail a numerical method for (1.3) and (1.4). Using, and at the same time generalizing, ideas contained implicitly in the paper [48] of HUBER (1939), he employs continuous, piecewise quadratic polynomials to approximate the solution to the given integral equation; this approximation is determined by so-called collocation techniques. While the application of this method to Volterra integral equations arising in heat conduction problems is given particular attention, there is no analysis of its convergence properties.

The subsequent development of numerical methods for Volterra integral equations possessing weakly singular kernels was based mainly on the work of YOUNG [96] of 1954 on product integration techniques. We refer the reader to the relevant references in [59] and [15] for additional details.

3.4. Conclusion

When surveying the contributions to the numerical solution of second-kind Volterra integral equations up to about 1965 one is perhaps struck by the fact that, with the possible exception of WHITTAKER [92] and MIKELADZE [65], they all deal with specific examples of methods and that a more unified view is still very much lacking. It seems interesting to observe that there emerges a rather different picture if one looks at the early development of numerical methods for Fredholm integral equations of the second-kind,

$$y(t) = g(t) + \lambda \int_0^T K(t,s)y(s)ds, \quad t \in I:$$

here, the two earliest methods, Bateman's method ([7]) of 1922 and Nyströms method ([71]) of 1928 represent very general approaches to generating numerical approximations to the solutions of such equations.

We conclude this section by mentioning that early surveys of numerical methods for Volterra and Fredholm integral equations (containing most, but not all, of the methods described here) may be found in BERNIER [10], FOX and GOODWIN [32], MAYERS [63], and NOBLE [69]. In addition, see also [85] and the extensive bibliography [70] by NOBLE.

4. RECENT DEVELOPMENTS

For the sake of completeness we name a few references to recent work on the approximate solution of second-kind Volterra equations.

(i) A very general analysis of quadrature methods (3.3) for (1.1) and (1.3) was recently given by WOLKENFELT [94] (see also his thesis of 1981). Generalizations of such methods are discussed in WOLKENFELT [95] and in VAN DER HOUWEN and TE RIELE [46], [47]. Compare also BRUNNER and VAN DER HOUWEN [19, Ch. 3]. Fractional quadrature methods for

equations with weakly singular kernels were introduced by LUBICH [62].

(ii) A particular class of implicit Runge-Kutta methods (3.4) was analyzed in detail by DE HOOG and WEISS [45]. The general Runge-Kutta theory for Volterra equations (1.3) with smooth kernels was established by BRUNNER, HAIRER and NØRSETT [18]; a comprehensive convergence analysis for general one-step methods (including Runge-Kutta methods of Pouzet and Bel'tyukov type) is due to HAIRER, LUBICH and NØRSETT [38]. LUBICH [61] extended the Runge-Kutta theory of [18] to weakly singular Volterra equations (1.4).

(iii) A recent account of collocation methods based on polynomial splines may be found in BRUNNER [14]; see also [16]. While these papers focus on Volterra equations with regular kernels, [15] and [17] are concerned with the problem of generating high-order approximations to (nonsmooth) solution of equations with weakly singular kernels. Compare also TE RIELE [78] where non-polynomial spline functions are employed to obtain such approximations.

(iv) Abstract convergence analysis (including numerous examples) of discretization methods for second-kind Volterra integral equations may be found in SCOTT [81] and in DIXON and MCKEE [28].

(v) There still exists only very few relatively general analyses of numerical stability of methods for second-kind Volterra equations. The two principal ones, dealing with convolution kernels, are those by LUBICH [60] (for quadrature methods) and by HAIRER and LUBICH [37] (for Runge-Kutta methods (3.4)). As regards equations with more general kernels, we refer to [19, Ch. 7]. The problem of numerical stability when solving weakly singular Volterra equations is still very much in the open; see, however, LUBICH [62].

(vi) The principal tool in the convergence analysis of numerical methods is the discrete Gronwall inequality,

$$z_n \leqslant C_1 h^{1-\alpha} \sum_{i=0}^{n-1} (n-i)^{-\alpha} z_i + C_2, \quad n=0,...,N, \ 0 \leqslant \alpha < 1 ,$$

where $z_i \geqslant 0$, $C_1 > 0$, $C_2 > 0$, and $Nh \leqslant c < \infty$. As mentioned in Section 3, the first occurrence of such an inequality (with $\alpha = 0$) seems to be in MIKELADZE [65, p. 259]. Of the more recent contributions in this area we mention the ones by JONES [50] and, especially, by BEESACK [8]. The case $0 < \alpha < 1$ is treated in detail in MCKEE [64], DIXON and MCKEE [27], and SCOTT [81].

(vii) Recent surveys of numerical methods for Volterra integral equations may be found in TE RIELE [77], BRUNNER [13], and BAKER [5]. The proceedings [24], [6] and [41] provide a good indication as to the current activities in the numerical analysis of Volterra equations. The first comprehensive monograph on the numerical treatment of Volterra (and Fredholm) integral equations, BAKER [4], is of quite recent origin: it appeared in 1977. More recent treatises are LINZ [57] and BRUNNER and VAN DER HOUWEN [19].

5. Automatic computer codes

The development of efficient and reliable software for second-kind Volterra integral equations is a very new activity (compare the comments in [30, p. 13]; see also Delves [23] and the article by Miller in [24, pp. 247-256]). At the time of writing, both the IMSL and NAG libraries did not contain any procedures for such equations. For historical reasons we mention the collection of ALGOL procedures [76] where one finds a number of non-automatic codes based on Pouzet's work on Runge-Kutta methods and Adams-type quadrature methods. In addition see also [88].

In the following we list some of the recently developed codes involving local or global error estimation and/or automatic stepsize change.

(i) The code of Bownds and Appelbaum [12] is based on kernel approximation techniques, the resulting integral equation is then equivalent to a system of (nonlinear) ordinary differential equations which are solved by a standard Adams or Runge-Kutta-Fehlberg code. See also the pertinent comments in [82] on the choice of the differential equation code if the system turns out to be stiff.

(ii) Codes using specific quadrature methods of the form (3.3) were written by Logan [59] (Simpson's method, with a block-by-block option); Hock [44] (midpoint method, followed by extrapolation techniques); Kunkel [55], Williams and McKee [93], and Jones [51] (predictor-corrector techniques). The only automatic code for weakly singular equations (1.4) with $\alpha = \frac{1}{2}$ is due to Logan [59] (product Simpson's method, used in block-by-block mode).

(iii) The following codes employ Runge-Kutta type methods: Tanfulla and Ribighini [86] (explicit, embedded Pouzet methods of orders 4 and 5); Duncan [29] (explicit, embedded 6-stage and 8-stage methods of Pouzet type and with orders 5 and 6); Schlichte [79] (implicit method of De Hoog and Weiss [45]); Hairer, Lubich and Schlichte [39] (explicit 4-stage Pouzet method of order 4, combined with fast Fourier transform techniques; this method is devised for equations with convolution kernels); and Blom and Brunner [11] (implicit Pouzet-type methods of variable orders, combined with discretized iterated collocation; the resulting local superconvergence properties are used to obtain error estimates). All of these codes are designed for Volterra integral equations possessing regular (bounded) kernels.

A more detailed description of the above codes may be bound in [19, Ch. 8].

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